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Matter-wave bright solitons: Internal atomic recombination and external feeding

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Abstract. We consider matter-wave bright solitons in the presence of three-body atomic recombination, an axial periodic modulation and a feeding term, and use a variational method to derive conditions to have dynamically stabilized solitons due to compensation between the dissipation and alimentation of atoms from external sources. We critically examine how the BEC soliton is affected by the imbalance between the internal atom loss and external feeding. We pay special attention to study the influence of these terms on the soliton dynamics in optical lattice potentials that cause periodic modulation.

PACS. 03.75.Lm Tunneling, Josephson effect, Bose-Einstein condensates in periodic potentials, solitons, vortices, and topological excitations – 05.45.Yv Solitons

1 Introduction

Almost simultaneously, but independently of each other, two different groups, one at École Normale Supérieure in Paris [1] and other at Rice University [2], observed matter-wave bright solitons by transferring a Bose-Einstein condensate (BEC) of ^{7}Li from the initial magnetic trap to an optical trap and ultimately releasing the BEC into a horizontal one-dimensional (1D) waveguide. The Paris experiment was carried out with a small number of atoms in the BEC and its motion was studied in an expulsive potential. In contrast with this, the Rice experiment produced a train of as many as fifteen solitons by working with a condensate consisting of a large number of atoms. These solitons — resulting from modulational instability (MI) [3] induced primarily by the manipulation of atomic scattering length — are apparently stabilized by repulsive soliton-soliton interactions. However, a soliton in a BEC can suffer from loss of atoms due to two- and three-body recombination. The two-body loss can be completely suppressed in the internal atomic state $|F=1, m_F=1\rangle$ of ^{7}Li obtained by magnetic tuning of the scattering length that parameterizes the two-body atom-atom interaction. But the three-body loss remains sizeable. The BEC dynamics may also be affected by supply of atoms from an external source to compensate the loss due to three-body atomic recombination.

In this work we try to accommodate the effect of dissipation and external feeding on the BEC within the framework of mean-field approximation in which the formation and subsequent evolution of the BEC solitons are governed by the time-dependent Gross-Pitaevskii (GP) equation written as

$$\left[-i\hbar\frac{\partial}{\partial t}-\frac{\hbar^2}{2m}\nabla^2+V_{ext}(\vec{r})+V_{int}(\vec{r})+\frac{i}{2}\Gamma\right]\psi=0. \quad (1)$$

Here $\psi=\psi(\vec{r},t)$ is the wave function of the condensate. The function ψ is also called the order parameter. The external potential is of the form

$$V_{ext}(\vec{r})=\frac{1}{2}m\omega_\perp^2\left(r^2+\lambda^2 z^2\right)+\kappa E_R\cos^2(k_L z) \quad (2)$$

with $r^2=x^2+y^2$. The first term in (2) represents a harmonic trap arising due to initial magnetic confinement of the BEC while the second term is a contribution to $V_{ext}(\vec{r})$ by the periodic optical lattice. We note that $\omega_\perp$ is the radial trapping frequency and $\lambda=\frac{\omega_z}{\omega_\perp}$ with ω_z, the axial trapping frequency. Clearly, m stands for the mass of each atom in the condensate. We have denoted the strength of the periodic potential by κ and wavenumber of the interference pattern in the optical lattice by k_L $\left(=\frac{2\pi}{\lambda}\right)$ such that the recoil energy $E_R=\frac{\hbar^2k_L^2}{2m}$ [4]. The internal potential is given by

$$V_{int}(\vec{r})=V_{HF}(\vec{r})+V_{3B}(\vec{r}), \quad (3)$$

where $V_{3B}(\vec{r})$ is the contribution to V_{int} due to three-body recombination. Understandably, $V_{HF}(\vec{r})$ is the usual Hartree-Fock potential as arises from the mean-field approximation. If the GP equation involves the interactions $V_{ext}(\vec{r})$ and $V_{int}(\vec{r})$ only, then the BEC solitons will decay and eventually disappear as they propagate. However, in addition to these interactions, if we can have a supply of

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atoms from external sources, we can produce dissipation managed solitons. The quantity $i\frac{\Gamma}{2}$ in (1) represents such an appropriate feeding term. The expressions for $V_{HF}(\vec{r})$ and $V_{3B}(\vec{r})$ are given by

$$V_{HF}(\vec{r}) = g|\psi|^2 \quad \text{and} \quad V_{3B}(\vec{r}) = -i\frac{\hbar}{2}K_3|\psi|^4. \tag{4}$$

The strength of the two-body interaction is $g = \frac{4\pi\hbar^2 a_s}{m}$ with a_s, the atomic scattering length. Here K_3 represents the rate of three-body recombination loss. Abdulleav and Salerno [5] investigated the general properties of localized states of 1D BECs in optical lattices with the elastic three-body inter-atomic interactions modeled by a real quintic nonlinearity. They examined at some length the effect of inelastic scattering by adding a quintic dissipative term just as ours. The general conclusion was that the damping effects are important only when the ratio between the imaginary and real parts of the three-body interaction is not small. Although, there has been some discussion on the effect of atoms feeding from the thermal cloud, consequences of the imbalance between feeding and three-body recombination terms remained largely unexplored. However, such studies are expected to be quite interesting. For example, in the recent past, Adhikari [6] used a perturbative procedure to show that alimentation of atoms from external sources may compensate the dissipation loss to produce a dynamically stabilized soliton.

The object of the present paper is to derive a useful theoretical framework for studying the properties of matter-wave bright solitons as governed by (1). We shall be concerned with a quasi-one-dimensional (Q1D) BEC soliton in a ciger-shaped trap with axially symmetric configuration. We shall first consider how a dynamically stabilized soliton could be prepared in this geometry and then examine, in some detail, the dynamics of BEC solitons formed on an optical lattice potential when there is imblance between the feeding and three-body recombination terms. The GP equation in (1) is nonintegrable. Ideally, therefore, one would like to implement a purely numerical routine to study the dynamical properties of the soliton solution supported by it. We shall, however, take recourse to the use of an approximation method and try to proceed analytically as far as possible and invoke the inevitable numerical programme only at the last step. The method we follow here will be based on a variational approximation which incorporates the width and amplitude as variational parameters. In this context, we note that, in addition to being nonintegrable, the equation under consideration represents a dissipative system. As opposed to conservative systems, the equation of motion of dissipative dynamical systems cannot be treated by the variational principle. This tends to pose an awkward analytical constraint for our proposed variational study. Over the years, a number of methods have been devised to circumvent this discrimination against nonconservative systems [7]. One of the best known methods is provided by the Rayleigh dissipation function [8]. In this case, it takes two scalar functions to specify the equation of motion. This route has been followed by Cerda et al. [9] to develop a variational approach to study nonlinear dissipative pulse propagation. The method is quite effective to deal with two- and three-body inelastic processes [10] that are present in a BEC.

In Section 2 we seek a Q1D reduction of (1) and briefly indicate how one could use a variational method to solve the reduced equation. We introduce, in Section 3, the explicit form of the trial wave function having a number of variational parameters and obtain the evolution equations of these parameters. Furthermore, we find an expression for the amplitude of the dissipation managed soliton in term of Γ and K_3. When the atom loss and the external feeding are unequal, the number of atoms in the BEC soliton becomes a function of time. The width and amplitude of the soliton also become time dependent. We derive equations for these time-dependent quantities with a view to compare in Section 4 the properties of overfed and dissipative solitons with those of the stabilized soliton.

2 Q1D GP equation and variational method

2.1 Dimensional reduction of (1)

To obtain the Q1D equation from (1) we begin by expressing it in the dimensionless variables defined by

$$\tau = \omega_\perp t, \ \rho = \frac{r}{a_0}, \ s = \frac{z}{a_0}, \ \ \psi(r,z,t) = \frac{u(\rho,s,\tau)}{a_0^{\frac{3}{2}}}.$$

The quantity $a_0 = \sqrt{\frac{\hbar}{m\omega_\perp}}$ stands for the size of the ground-state solution of the noninteracting GP equation. This gives

$$\begin{aligned} -iu_\tau - \frac{1}{2}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{\partial^2 u}{\partial s^2}\right] + \frac{1}{2}\left(\rho^2 + \lambda^2 s^2\right)u \\ + \frac{\kappa E_R \cos^2(k_L a_0 s)}{\hbar\omega_\perp}u + \frac{i}{2}\frac{\Gamma}{\hbar\omega_\perp}u + \frac{4\pi\hbar a_s}{\omega_\perp a_0^3 m}|u|^2 u \\ - \frac{iK_3}{2\omega_\perp a_0^6}|u|^4 u = 0. \end{aligned} \tag{5a}$$

We have written (5a) in cylindrical coordinates. If the wave function $\psi(\vec{r},t)$ is normalized to the number of particles $N(t)$ in the condensate at any instant of time according to

$$\int |\psi|^2 d^3r = N(t) \tag{5b}$$

then it is obvious that

$$\int |u|^2 \rho d\rho ds = N(\tau)/2\pi. \tag{5c}$$

In writing (5b) and (5c) we have regarded the number of particles in the BEC as a function of time presumably because our system is either dissipative or overfed. Using a separable ansatz

$$u\left(\rho,\, s,\, \tau\right) = \phi(\rho)\xi(s,\, \tau) \tag{6}$$

we rewrite (5a) in the form

$$\frac{1}{\xi}\left[-i\xi_\tau - \frac{1}{2}\xi_{2s} + \frac{1}{2}\lambda^2 s^2 \xi + \frac{\kappa E_R \cos^2(k_L a_0 s)}{\hbar\omega_\perp}\xi + \frac{i}{2}\frac{\Gamma}{\hbar\omega_\perp}\xi\right] + 4\pi\frac{a_s}{a_0}|\xi|^2|\phi|^2 - \frac{iK_3}{2\omega_\perp a_0^6}|\xi|^4|\xi|^4 = -\frac{1}{\phi(\rho)}\left[-\frac{1}{2}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{2}\rho^2\phi\right]. \quad (7)$$

In (7) the subscript on ξ stands for partial derivatives with respect to that particular independent variable. More specifically, $\xi_{2s} = \frac{\partial^2\xi}{\partial s^2}$. This equation shows that the presence of atom-atom interaction and three-body recombination loss does not permit clear cut separation of variables. However, these terms are quite small such that ϕ may be assumed to satisfy

$$-\frac{1}{2}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{2}\rho^2\phi = \omega_\rho\phi \quad (8)$$

with ω_ρ related to $\omega_\perp$ by a scaling factor. Equation (8) represents the well-known eigenvalue problem for the two-dimensional harmonic oscillator with the ground state solution given by

$$\phi_0(\rho) = e^{-\rho^2/2}. \quad (9)$$

Thus (7) can be written as

$$-i\xi_\tau - \frac{1}{2}\xi_{2s} + \frac{1}{2}\lambda^2 s^2\xi + \frac{\kappa E_R\cos^2(k_L a_0 s)}{\hbar\omega_\perp}\xi + \frac{i}{2}\frac{\Gamma}{\hbar\omega_\perp}\xi + 4\pi\frac{a_s}{a_0}|\xi|^2\xi|\phi|^2 - \frac{iK_3}{2\omega_\perp a_0^6}|\xi|^4\xi|\phi|^4 = \omega_\rho\xi. \quad (10)$$

The low-frequency vibration along the z-direction is quite unlikely to excite the ground state. In view of this we multiply (10) by $\phi^\star\phi$ and integrate over the ρ coordinate to get

$$-i\xi_\tau - \frac{1}{2}\xi_{2s} + \frac{1}{2}\lambda^2 s^2\xi + \frac{\kappa E_R\cos^2(k_L a_0 s)}{\hbar\omega_\perp}\xi + \frac{i}{2}\frac{\Gamma}{\hbar\omega_\perp}\xi + 2\pi\frac{a_s}{a_0}|\xi|^2\xi - \frac{iK_3}{6\omega_\perp a_0^6}|\xi|^4\xi = \omega_\rho\xi. \quad (11)$$

Equation (11) can be written in a more convenient form by using the change of variable

$$\xi(s,\tau) = \chi(s,\tau)e^{-i\omega_\rho\tau}. \quad (12)$$

Using (12) in (10) we write

$$i\chi_\tau + \frac{1}{2}\chi_{2s} + \frac{1}{2}\lambda_z^2 s^2\chi - \kappa E_R\cos^2(k_L a_0 s)\chi - \frac{2\pi a_s}{a_0}|\chi|^2\chi = i\left(\frac{\Gamma}{2\hbar\omega_\perp}\chi - \frac{K_3}{6\omega_\perp a_0^6}|\chi|^4\chi\right) \quad (13)$$

with

$$\int_{-\infty}^{+\infty}|\chi|^2 ds = \frac{N}{\pi}. \quad (14)$$

Equation (13) represents our desired Q1D equation. Because of the constant supply of atoms we will be able to derive conditions under which the solution of (13) will support a dissipation managed soliton solution with an axial periodic modulation. In the following we outline how our initial boundary value problem can be converted into a variational problem.

2.2 Variational formulation of (13)

If the right side of (13) is zero the corresponding initial boundary-value problem can be converted to a variational problem with a Lagrangian density written as

$$\mathcal{L}_C = i\chi^\star\chi_\tau - \frac{1}{2}\chi_s\chi_s^\star - \frac{1}{2}\lambda^2 s^2\chi\chi^\star - \frac{\kappa E_R\cos^2(k_L a_0 s)}{\hbar\omega_\perp}\chi\chi^\star - 2\frac{\pi a_s}{a_0}\chi^2\chi^{\star 2}. \quad (15)$$

We have used the subscript C on $\mathcal{L}$ to indicate that the Lagrangian density in (15) when substituted in the appropriate Euler-Lagrangian equation gives only the conservative part of (13). To incorporate the effects of dissipation and of feeding we must now add to $\mathcal{L}_C$ a term $\mathcal{L}_{NC}$ closely related to the Rayleigh dissipation function such that

$$\frac{\delta\mathcal{L}_{NC}}{\delta\chi^\star} = -R(\chi,\chi^\star) \quad (16)$$

with

$$R(\chi,\chi^\star) = -i\left(\frac{\Gamma}{2\hbar\omega_\perp}\chi - \frac{K_3}{6\omega_\perp a_0^6}|\chi|^4\chi\right). \quad (17)$$

If we now introduce an appropriate trial function for $\chi(s,\tau)$ characterized by the variational parameter η_i, then the Rayleigh-Ritz optimization procedure when applied to the Euler-Lagrange equation for the total Lagrangian will lead to system of equations for η_i. In particular, we shall get [10]

$$\frac{\partial L}{\partial\eta_i} - \frac{d}{dt}\frac{\partial L}{\partial\eta_{it}} = \int ds\left[R\frac{\partial\chi^\star}{\partial\eta_i} + R^\star\frac{\partial\chi}{\partial\eta_i}\right], \quad (18)$$

where

$$L = \int\mathcal{L}_C^t ds. \quad (19)$$

The superscript t on $\mathcal{L}_C$ indicates that before carrying out the integral in (19) we must write the conservative Lagrangian in terms of a trial wave function.

3 Variational equations for the parameters in the trial function

To implement (18) for studying the properties of BEC solitons which evolve according to (13) we shall make use of the Gaussian trial function

$$\chi(s,\tau) = A(\tau)\exp\left[-\frac{s^2}{2a(\tau)^2} + \frac{i}{2}b(\tau)s^2 + i\phi(\tau)\right], \quad (20)$$

where A, a, b and ϕ are the amplitude, width, frequency chirp and linear phase. Note that in writing (18) the parameters A, a, b and ϕ were collectively denoted by η_i. Since s is dimensionless, $a(\tau)$ must also represent a dimensionless quantity. The choice (20) is convenient but by no means the only possible one, e.g. a hyperbolic secant shaped ansatz would have done equally well. For the wave function in (20) the normalization condition in (14) gives

$$a(\tau)A(\tau)^2 = \frac{N(\tau)}{\pi^{3/2}} \tag{21}$$

for all values of τ.

From (15), (19) and (20) we get

$$L = \sqrt{\pi}\Bigg[iA_\tau Aa - A^2 a\phi_\tau + \frac{i}{2}a_\tau A^2 - \frac{1}{4}A^2 b_\tau a^3 - \frac{1}{4}\left(\frac{1}{a} + b^2a^3\right)A^2 - \frac{1}{4}\lambda^2 a^3 A^2 - \frac{\kappa E_R}{\hbar\omega_\perp}\left(a + ae^{-k_L^2 a_0^2 a^2}\right)A^2 - \frac{\pi}{\sqrt{2}}\frac{a_s}{a_0}aA^4\Bigg]. \tag{22}$$

The reduced Lagrangian in (22) in conjunction with (18) yields the following equations.

$$2Aa\phi_\tau + \frac{1}{2}Ab_\tau a^3 + \frac{1}{2}\left(\frac{1}{a} + b^2a^3\right)A + \frac{1}{2}\lambda^2 a^3 A + \frac{2\kappa E_R}{\hbar\omega_\perp}\left(a + ae^{-k_L^2 a_0^2 a^2}\right)A^2 + \frac{4\pi}{\sqrt{2}}\frac{a_s}{a_0}aA^3 = 0, \tag{23}$$

$$A^2\phi_\tau + \frac{3}{4}A^2 b_\tau a^2 + \frac{1}{4}\left(-\frac{1}{a^2} + 3b^2a^2\right)A^2 + \frac{3}{4}\lambda^2 a^2 A^2 + \frac{\kappa E_R}{\hbar\omega_\perp}\left(1 + e^{-k_L^2 a^2 a_0^2} - 2k_L^2 a_0^2 a^2 e^{-k_L^2 a^2 a_0^2}\right)A^2 + \frac{\pi}{\sqrt{2}}\frac{a_s}{a_0}A^4 = 0, \tag{24}$$

$$\frac{1}{2}\frac{d}{d\tau}\left(a^3A^2\right) - A^2ba^3 = \left[\frac{\Gamma}{2\hbar\omega_\perp}a^3 - \frac{K_3A^4}{6\omega_\perp a_0^3}\frac{a^3}{3^{3/2}}\right]A^2 \tag{25}$$

and

$$\frac{d}{d\tau}\left(aA^2\right) = 2A^2\left[\frac{\Gamma}{2\hbar\omega_\perp}a - \frac{K_3A^4}{6\omega_\perp a_0^3}\frac{a}{\sqrt{3}}\right]. \tag{26}$$

From (21) and (26) we have

$$\frac{dN}{d\tau} = 2N\left[\frac{\Gamma}{2\hbar\omega_\perp} - \frac{K_3A^4}{6\sqrt{3}\omega_\perp a_0^6}\right]. \tag{27}$$

A dissipation managed soliton is obtained when $\frac{dN}{d\tau} = 0$. In this case (27) yields a result for the amplitude given by

$$A = \left(\frac{\Gamma}{\hbar\omega_\perp}\frac{3\sqrt{3}\omega_\perp a_0^6}{K_3}\right)^{1/4}. \tag{28}$$

The expression in (28) is in agreement with that obtained using a perturbation theory method [6]. Using (21) in (27) we can write the loss-rate equation in a slightly different form

$$\frac{1}{N}\frac{dN}{d\tau} = \left[\frac{\Gamma}{\hbar\omega_\perp} - \frac{K_3}{3\sqrt{3}\omega_\perp a_0^6\pi^3}\frac{N^2}{a^2}\right]. \tag{29}$$

Equations (23) and (24) can be combined to get

$$-A^2b_\tau a^3 + \frac{A^2}{a} - a^3b^2A^2 - \lambda^2a^3A^2 + \sqrt{2}\pi\frac{a_s}{a_0}aA^4 + \frac{4\kappa E_R}{\hbar\omega_\perp}a_0^2k_L^2a^2e^{-a_0^2k_L^2a^2}aA^2 = 0. \tag{30}$$

Similarly, from (25) and (26) we have

$$ab - \frac{da}{d\tau} = -\frac{K_3aA^4}{9\sqrt{3}\omega_\perp a_0^6}\cdot \tag{31}$$

Eliminating b from (30) and (31) we get an ordinary differential equation for $a(\tau)$ given by

$$a_{2\tau} - \frac{2K_3NN_\tau}{9\sqrt{3}\pi^3\omega a_0^6}\frac{1}{a} + \frac{K_3^2N^4}{\left(9\sqrt{3}\pi^3\omega_\perp a_0^6\right)^2}\frac{1}{a^3} - \frac{1}{a^3} + \lambda^2 a - \sqrt{\frac{2}{\pi}}\frac{Na_s}{a_0}\frac{1}{a^2} - \frac{4\kappa E_R}{\hbar\omega_\perp}a_0^2k_L^2ae^{-a_0^2k_L^2a^2} = 0. \tag{32}$$

Clearly, (29) and (32) represent a set of coupled equations which we need to solve simultaneously to get the values of $N(\tau)$ and $a(\tau)$. Note that the differential equation for $N(\tau)$ is of first order while that for $a(\tau)$ is of second order. This poses a problem to solve these equations simultaneously which one needs to study the properties of dissipative solitons. To circumvent this difficulty we venture to suggest that replacement of $a(\tau)^2$ by $\langle a(0)^2\rangle$ in (29) will not fare worse in the investigation of such properties in the presence of three-body atomic recombination. Khaykovich et al. [1] quotes the experimental result for $\langle a(0)\rangle$ as 10 μm. We shall work with this value of $\langle a(0)\rangle$ to express $N(\tau)$ as a function of τ. Since $s = \frac{z}{a_0}$ we write $\langle a^2(0)\rangle = (\frac{10}{a_0})^2$ with a_0, the transverse oscillator length measured in micrometer. In our case $a_0 = 1.4256$ μm for $\omega_\perp = 2\pi \times 710$ Hz [1] such that $\langle a^2(0)\rangle = 49.2065$.

With this average value for $a(\tau)$ we solve (29) to get

$$N(\tau) = \left[\frac{C_2}{\frac{3C_1}{49.2065} + \left(\frac{C_2}{N_0^2} - \frac{3C_1}{49.2065}\right)e^{-2C_2\tau}}\right]^{1/2}, \tag{33}$$

where

$$C_1 = \frac{K_3}{9\sqrt{3}\pi^3\omega_\perp a_0^6} \quad \text{and} \quad C_2 = \frac{\Gamma}{\hbar\omega_\perp}\cdot \tag{34}$$

Clearly, the constant of integration N_0 represents the initial number of atoms in the condensate. For a given value of N_0, we can use (33) to investigate how the imbalance between the feeding and dissipative terms affects the loss or gain of atoms from BEC. For $\tau = 0$, (33) gives $N(0) = N_0$ while for $\tau = \infty$ this equation gives $N(\infty) = \left(\frac{49.2065}{3}\frac{C_2}{C_1}\right)^{\frac{1}{2}}$. Understandably, for dissipative

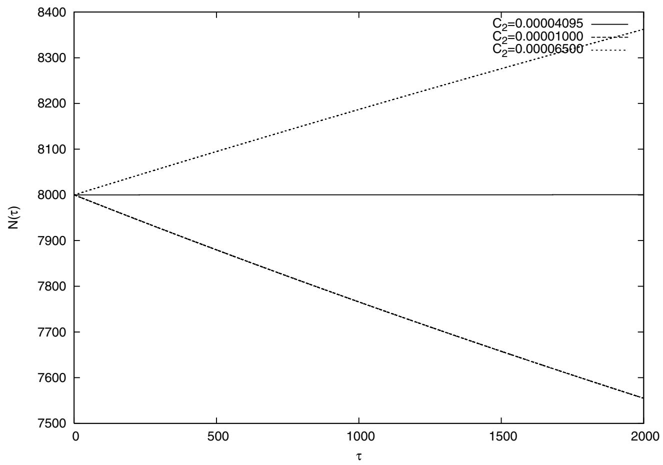

Fig. 1. Number of atoms $N(\tau)$ as a function of τ from (33).

solitons $N(\infty) \ll N_0$. This inequality sets a criterion for the choice of C_2 and we have $C_2 \ll 0.06C_1N_0^2$. For a suitable value of C_2, $N(\tau)$ from (33) can be substituted in (32) to obtain $a(\tau)$ as a function of τ. The results for $a(\tau)$ and $N(\tau)$ in conjunction with (21) determine $A(\tau)$ for complete specification of the density profile $|\chi(s,\tau)|^2$.

4 Spatio-temporal behaviour of BEC solitons

We shall first study how, in the presence of three-body atomic recombination and external feeding, the number of atoms in the BEC changes with τ. To that end we have chosen to work with $K_3 = 1.9 \times 10^{-26}$ cm^6/s and $\omega_\perp = 2\pi \times 710$ Hz [1] for which, as we have seen, transverse oscillator length $a_0 = 1.426 \times 10^{-4}$ cm. The result of K_3 quoted here is for ^{7}Li. The values of K_3 change significantly if we consider BEC of other atomic gases [11]. The values used by us for other parameters in (32) are $\kappa = 0.2$ and $a_s = -1.59 \times 10^{-7}$ cm^{-1} [12]. There are three cases of interest. First the feeding term may dominate over the atom loss due to three-body recombination. Secondly, there may be perfect balance between these two effects. Finally, the internal atom loss may dominate over the external feeding. In Figure 1 we plot $N(\tau)$ as a function of τ as calculated from (33). Corresponding to these three physical situations we have used $C_2 = 6.500 \times 10^{-5}$, $C_2 = 4.095 \times 10^{-5}$ and $C_2 = 1.000 \times 10^{-5}$ respectively for $N_0 = 8000$. As expected, in the first case $N(\tau)$ increases with τ (dotted line), in the second case $N(\tau)$ is a constant for all τ being equal to N_0 (solid line) and finally, in the third case $N(\tau)$ decreases with τ (dashed line).

Substituting the values of $N(\tau)$ and $N_\tau(\tau)$ from (33) in (32) one can, in principle, make use of the fourth-order Runge-Kutta method to solve the latter with the initial conditions $a(0) = 7.0149$ and $\frac{da(\tau)}{d\tau}|_{\tau=0} = 0$. For given values of $a(\tau)$ and $N(\tau)$, the corresponding results for $A(\tau)$ can be obtained from (21). In this context we observe that due to the transformation of length and the time scales sought by us, we shall not be able to study the effect of lattice potential even for very small values of κ_L. This is apparent from the last term of (32). To study the effect of an optical lattice on the BEC condensate one should ideally, follow Choi and Niu [13] to rescale (1). In fact, Abdullaev and Salerno [5] used this rescaling in their study of gap-Townes solitons in the presence of dissipation. However, the choice of our scale change does not preclude the possibility for including the effect of optical lattice on the BEC dynamics provided we agree to work with values of κ_L close to $1/\sqrt{a(0)}a_0 = 2.645 \times 10^3$ cm^{-1}. We have, therefore, chosen to work with these typical values of κ_L. Using our chosen values of C_2's we have plotted in Figure 2 the curves for $a(\tau)$ as a function of τ. The left (right) panel of this figure shows the variation of $a(\tau)$ in the absence (presence) of the lattice potential. Looking closely into this figure we see that, irrespective of the C_2 values, $a(\tau)$ as a function of τ exhibits the characteristic oscillatory behaviour [14]. When the atom loss dominates over feeding ($C_2 = 1.000 \times 10^{-5}$, dashed curve) the peak values of $a(\tau)$ for all τ lie above those of the stabilized soliton ($C_2 = 4.095 \times 10^{-5}$, solid curve). On the contrary, in the case of an overfed soliton ($C_2 = 6.500 \times 10^{-5}$, dotted curve) we observe the opposite. Furthermore, we note that the peaks in $a(\tau)$ of the overfed soliton lie to the left of the stabilized soliton and those of the underfed one lie to the right. This general behaviour of the curves does not change from the left to the right panel. In other words, the optical potential does not effect the nature of response of $a(\tau)$ to gain or loss of atoms by the BEC soliton. However, comparing the curves in the two panels, we observe that in the presence of the lattice potential, the cusp-like behaviour at the minima of $a(\tau)$ is smoothed out so as to produce typical sinusoidal behaviour.

Figure 3 shows the variation of $A(\tau)$ with τ. In the left panel, where the effect of the lattice potential is not taken into consideration, the peaks of $A(\tau)$ are very sharp. The peak values for the dissipative soliton ($C_2 = 1.000 \times 10^{-5}$, dotted curve) fall below those of the dissipation-managed soliton, while similar peak values for the overfed soliton ($C_2 = 6.500 \times 10^{-5}$, dotted curve) are considerably augmented. These observations on $a(\tau)$ and $A(\tau)$ are in agreement with the constraint implied by (21). It is evident from the figure in the right panel that the lattice potential smoothens out the sharp peaks and produces once again a sinusoidal pattern, although the nature of the response of $A(\tau)$ to the gain or loss of atoms remains unaltered.

In Figure 4 we display the density profile of the stabilized soliton ($C_2 = 4.095 \times 10^{-4}$). In the left panel (no optical lattice), the plane over the profile clearly exhibits that as time goes on, the soliton propagates without any dissipation. The same is true for the figure in the right panel where we include the effect of the optical potential. It is interesting to study Bose-Einstein condensates in optical lattices because here one adds a new length scale to the system, namely, the lattice spacing which is typically much smaller than the BEC itself. In addition to the harmonic confinement, the lattice potential introduces periodicity into the system and the new length scale leads to very large local trapping frequencies. In the limit

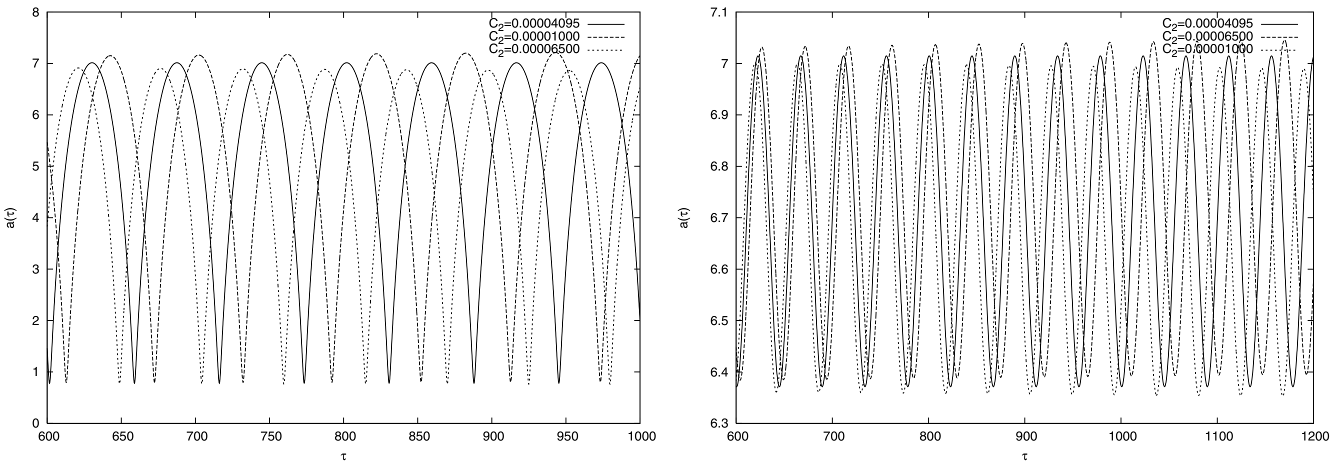

Fig. 2. Variation of Gaussian width $a(\tau)$ of the BEC soliton in the absence of optical lattice (left panel) and in the presence of optical lattice (right panel).

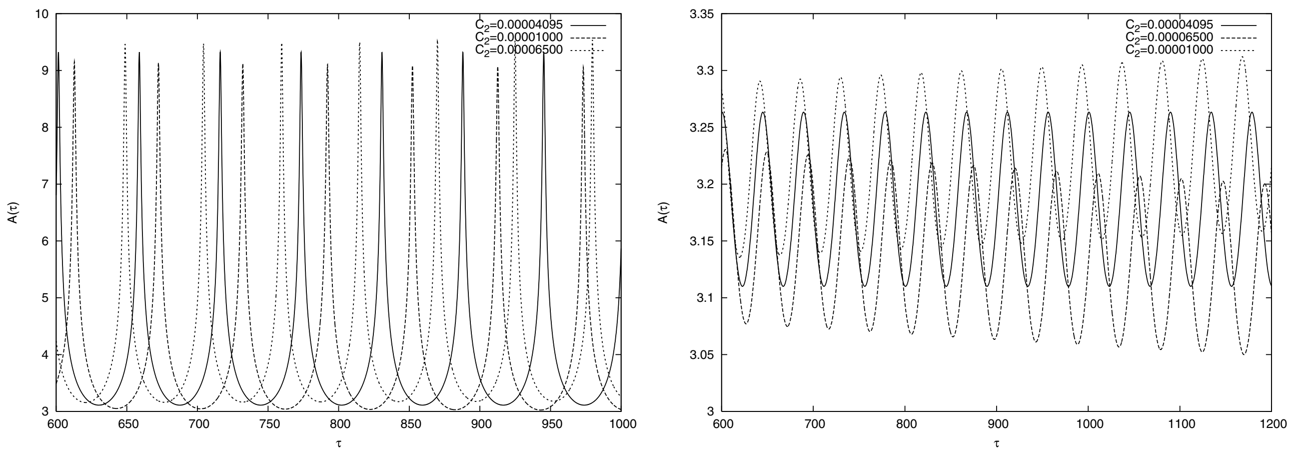

Fig. 3. Gaussian amplitude $A(\tau)$ of the BEC soliton as a function of τ. As in Figure 2 the left panel shows the variation of $A(\tau)$ in the absence of an optical lattice while the right panel gives a similar variation in the presence of an optical lattice.

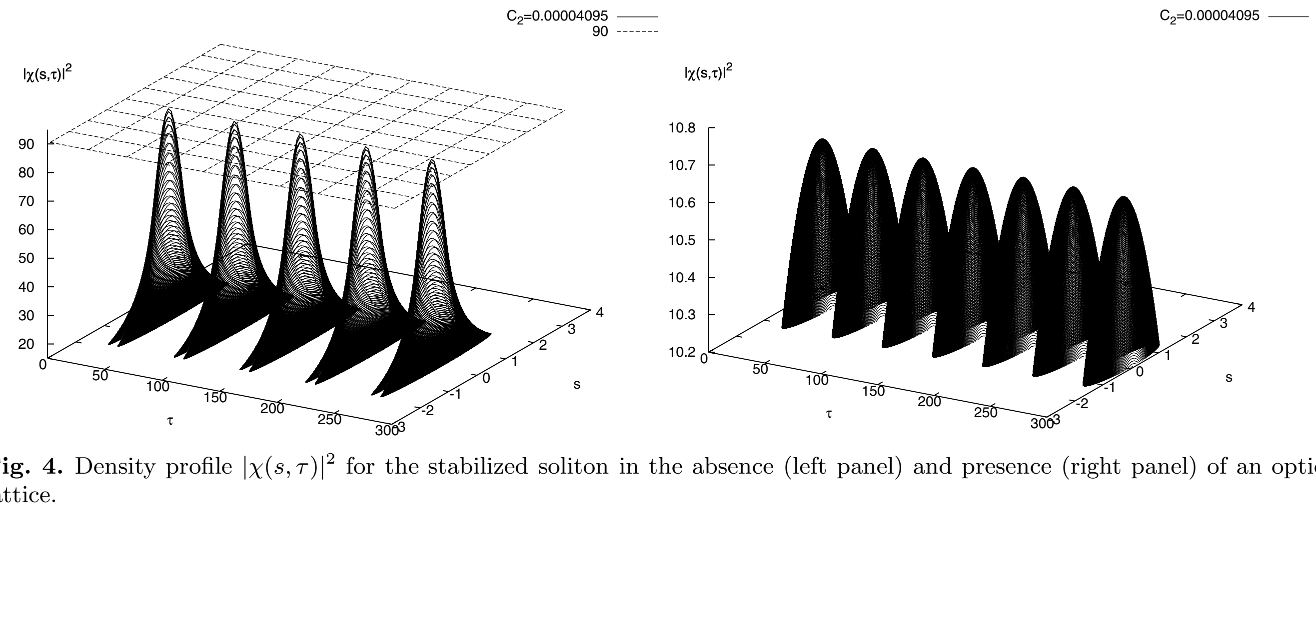

Fig. 4. Density profile $|\chi(s,\tau)|^2$ for the stabilized soliton in the absence (left panel) and presence (right panel) of an optical lattice.

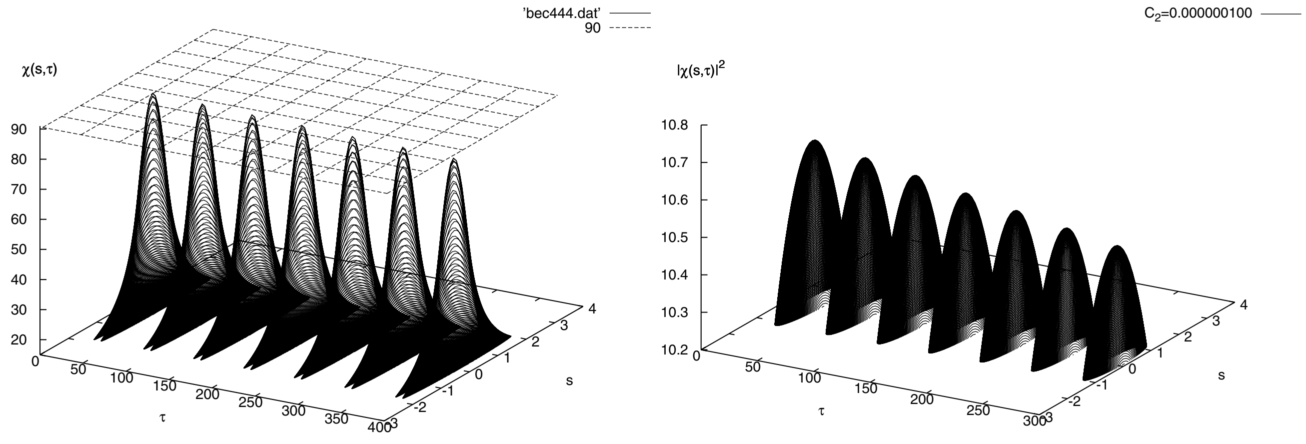

Fig. 5. Density profile $|\chi(s,\tau)|^2$ for the underfed or dissipative soliton. As in Figure 4 the profile values do not involve (left panel) and involve (right panel) the effect of an optical lattice.

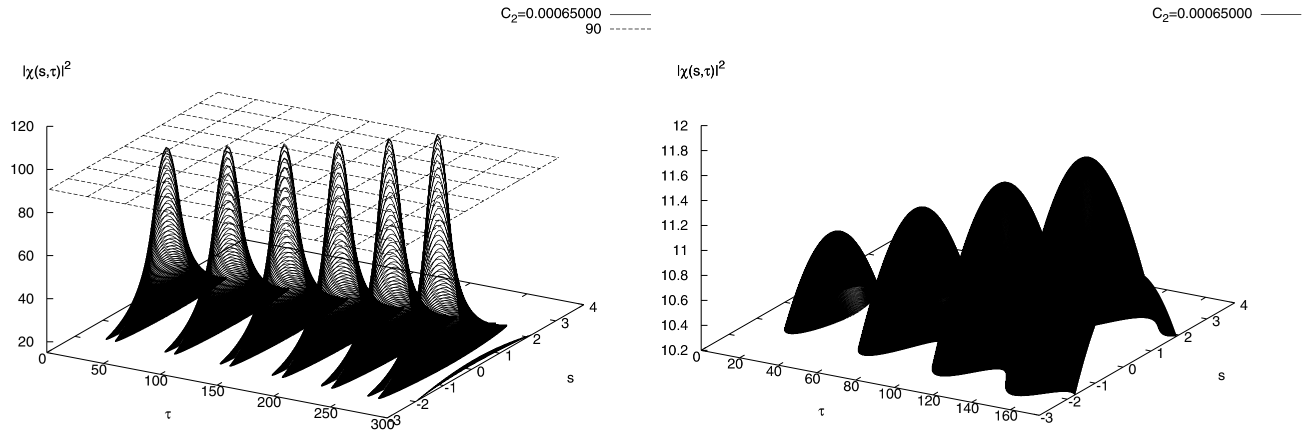

Fig. 6. Similar density profile $|\chi(s,\tau)|^2$ for the overfed soliton as in Figures 3 and 4.

of large lattice depths it is possible to have completely isolated micro condensates [4]. We note that in a theoretical framework, this length scale can be achieved by a change of variables as used by Choi and Niu [13]. However, we worked with a rescaling in which the lattice spacing and the size of the condensate are of the same order. We observe that, even in this case, the effect of the optical lattice is quite interesting. For example, the lattice potential squeezes the density profile (right panel) which again propagates without any dissipation. In Figures 5 and 6 we portray the soliton profiles for dissipative and overfed solitons. In both cases we see that the lattice potential produces a squeezing of the corresponding profiles. The height of the dissipative soliton decreases with τ while that for overfed soliton increases with τ. It is of interest to study the time evaluation of the BEC soliton at a particular value of s with a view to examine how the feeding term balances the damping and stabilizes the localized states against decay [5]. In view of this, we plot in Figure 7 $|\chi(1,\tau)|^2$ as a function of τ. In the balanced condition, the 2D profile values (solid curve) do not change with τ. The dissipative soliton profile (dashed curve) decays with τ while the profile for overfed soliton (dotted curve) grows as τ increases. We conclude by noting the following.

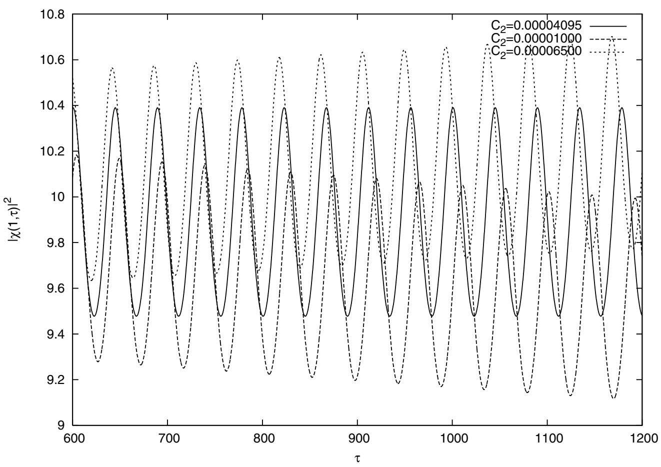

Fig. 7. The magnitude of $|\chi(s,\tau)|^2$ at $s = 1$ for different values of the feeding term is plotted as a function of τ.

The use of variational methods to construct solutions of nonlinear evolution equations is very effective when these equations follow from the action principle. The approach tends to break down for non-Lagrangian dynamical systems. Matter-wave bright solitons in the presence of three-body atomic recombination, an optical lattice and a phenomenological gain term belong to the latter class. This motivated us to use the method of Cadrá et al. [9] to study the perturbation of BEC solitons due to gain or loss of atoms. The significance of the final result was obtained by solving the equation for $N(\tau)$ to within an approximation. This limitation can be removed by converting the first-order equation for $N(\tau)$ by differentiating it once with respect to τ and then using the routine Runge-Kutta method to solve the coupled differential equations of $N(\tau)$ and $a(\tau)$. In that case one will require the value of $\frac{dN(\tau)}{d\tau}|_{\tau=0}$ which is not known before hand. However, we believe that our results are significant enough to initiate further works along this line of investigation.

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